Zezé's contributions on expansiveness

José Vieitez

Centro Universitario Regional Litoral Norte
Universidad de la República, Uruguay

Celebrating the $60 + 10$-aniversary of María José Pacifico

April 28, 2022

Zezé's interests in different aspects of mathematics - and beyond Mathematics - are so wide that we may apply to her Terencio's phrase: "nothing human is alien to me[1]".

[1] "*Homo sum, humani nihil a me alienum puto*"

Zezé's interests in different aspects of mathematics - and beyond Mathematics - are so wide that we may apply to her Terencio's phrase: "nothing human is alien to me[1]".

I wish to recall that when Ricardo Mañé was seriously ill in Montevideo, at the end of his life (1995), Zezé traveled from Rio to visit him. This says something about her quality as a human being.

[1] "*Homo sum, humani nihil a me alienum puto*"

Zezé's interests in different aspects of mathematics - and beyond Mathematics - are so wide that we may apply to her Terencio's phrase: "nothing human is alien to me[1]".

I wish to recall that when Ricardo Mañé was seriously ill in Montevideo, at the end of his life (1995), Zezé traveled from Rio to visit him. This says something about her quality as a human being.

But I have to concentrate just on her contributions on different aspects of expansiveness ...

[1] "*Homo sum, humani nihil a me alienum puto*"

Robust expansiveness: diffeomorphisms

Let (M, d) be a metric space and $f : M \to M$ an homeomorphism.

- define $\Gamma_\alpha(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \alpha,\ n \in \mathbb{Z}\}$.
  (points whose f-orbits α-shadow the f-orbit of x)

Robust expansiveness: diffeomorphisms

Let (M, d) be a metric space and $f : M \to M$ an homeomorphism.

- define $\Gamma_\alpha(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \alpha,\ n \in \mathbb{Z}\}$.
  (points whose f-orbits α-shadow the f-orbit of x)
- f is expansive iff $\exists\, \alpha > 0$ such that

$$\forall\, x \in M, \quad \Gamma_\alpha(x) = \{x\}$$

i.e., if $\forall n \in \mathbb{Z}$: $d(f^n(x), f^n(y)) \leq \alpha$ then $x = y$.

Robust expansiveness: diffeomorphisms

Let (M, d) be a metric space and $f : M \to M$ an homeomorphism.

- define $\Gamma_\alpha(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \alpha,\ n \in \mathbb{Z}\}$.
  (points whose f-orbits α-shadow the f-orbit of x)
- f is expansive iff $\exists\, \alpha > 0$ such that

$$\forall\, x \in M,\ \ \Gamma_\alpha(x) = \{x\}$$

i.e., if $\forall n \in \mathbb{Z}$: $d(f^n(x), f^n(y)) \leq \alpha$ then $x = y$.

First, robust expansiveness on the whole space:

Definition

Let M be a C^s closed manifold. $f : M \to M$ is C^r-robust expansive, $r \leq s$, if there is a C^r-neighborhood $\mathcal{N}$ of f in $\mathrm{Diff}^r(M)$ such that for any $g \in \mathcal{N}$, g is expansive.

C^1-robust expansiveness implies hyperbolicity

For $r \geq 2$, due to the lack of a general Closing Lemma, Connecting Lemma or Franks Lemma it is difficult to obtain results from C^r-robust expansivity.
So efforts were concentrated on C^1-robust expansiveness.

C^1-robust expansiveness implies hyperbolicity

For $r \geq 2$, due to the lack of a general Closing Lemma, Connecting Lemma or Franks Lemma it is difficult to obtain results from C^r-robust expansivity.
So efforts were concentrated on C^1-robust expansiveness.

Mañé studied the case in which for $f : M \to M$ there is a C^1-neighborhood $\mathcal{U}$ of f such that for $g \in \mathcal{U}$, g is expansive on the *whole manifold* (see R. Mañé *Expansive diffeomorphisms* Lectures Notes in Mathematics, Springer 468). He proved that in that case f is quasi-Anosov, that is: if $0 \neq v \in TM$ then $\|Df^n(v)\| \to \infty$ either for $n \to +\infty$ or for $n \to -\infty$) and *all the homoclinic classes are hyperbolic*.

Robust expansivity just in a homoclinic class.

Second,
robust expansiveness just on a part of the ambient manifold.

Robust expansivity just in a homoclinic class.

Second,
robust expansiveness just on a part of the ambient manifold.

Pacifico and Pujals proposed to study the case where we have a homoclinic class $H(p)$ of a diffeomorphism $f : M \to M$, where p is f-periodic, and in which f is C^1-robustly expansive only in $H(p)$.

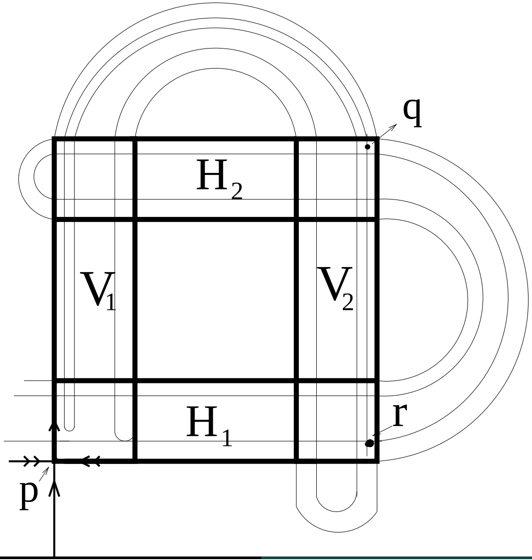
Robust expansivity just in a homoclinic class.

Second,
robust expansiveness just on a part of the ambient manifold.

Pacifico and Pujals proposed to study the case where we have a homoclinic class $H(p)$ of a diffeomorphism $f : M \to M$, where p is f-periodic, and in which f is C^1-robustly expansive only in $H(p)$.

But, what does it mean?
A homoclinic class may disappear, or more than one homoclinic class can be created, by means of a tiny perturbation.

p a saddle-node fixed point.



Definition

$H(p)$ is C^r-robustly expansive ($r \geq 1$) iff there exist $\alpha > 0$ and a C^r-neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exists a continuation p_g of p such that g is expansive in $H(p_g)$.

In order to prevent a "ghost" homoclinic class $H(p)$ after a perturbation, we assume that p is hyperbolic.

Definition

$H(p)$ is C^r-robustly expansive ($r \geq 1$) iff there exist $\alpha > 0$ and a C^r-neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exists a continuation p_g of p such that g is expansive in $H(p_g)$.

In order to prevent a "ghost" homoclinic class $H(p)$ after a perturbation, we assume that p is hyperbolic.

Let us recall the definition of a dominated splitting.

Definition

We say that a compact f-invariant set Λ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \;\forall x \in \Lambda,\; n \geq 0. \quad (1)$$

Results obtained with Zezé

As a first step we prove

Theorem (Pacifico, Pujals, V., *Erg. Th. & Dynam. Sys.* 25)

Let $dim(M) = 3$, $f \in \mathrm{Diff}^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is robustly expansive. Then for an open an dense subset $\mathcal{N}$ of a neighborhood $\mathcal{U}(f)$ in the C^1-topology, it holds that if $g \in \mathcal{N}$ then $H(p_g)$ has a co-dimension one dominated splitting $E \oplus F$.

Results obtained with Zezé

As a first step we prove

Theorem (Pacifico, Pujals, V., *Erg. Th. & Dynam. Sys.* 25)

Let $dim(M) = 3$, $f \in \mathrm{Diff}^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is robustly expansive. Then for an open an dense subset $\mathcal{N}$ of a neighborhood $\mathcal{U}(f)$ in the C^1-topology, it holds that if $g \in \mathcal{N}$ then $H(p_g)$ has a co-dimension one dominated splitting $E \oplus F$.

Theorem (Pacifico, Pujals, V., *Erg. Th. & Dynam. Sys.* 25)

Let $f \in \mathrm{Diff}^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is robustly expansive and has a co-dimension one dominated splitting $E \oplus F$. Then for an open an dense subset $\mathcal{N}$ of a neighborhood $\mathcal{U}(f)$ in the C^1-topology, it holds that if $g \in \mathcal{N}$ then $H(p_g)$ is hyperbolic.

Removing "open-denseness"

In fact C^1-robust expansiveness implies hyperbolicity. This was proved, for codimension one homoclinic classes:

Theorem (Pacifico, Pujals, M. Sambarino, V.)

Robustly expansive codimension-one homoclinic classes have a codimension-one dominated splitting $E \oplus F$.

Theorem (Pacifico, Pujals, M. Sambarino, V.)

Robustly expansive homoclinic classes with a codimension-one dominated splitting are hyperbolic.

These results appear in *Ergodic Theory Dynam. Systems* **29**.

expansiveness for flows

Zezé also contributed to the study of expansiveness on flows. Let me recall the definition given by Komuro.
Denote by $S(\mathbb{R})$ the set of surjective increasing continuous functions $h : \mathbb{R} \to \mathbb{R}$.

Definition (Komuro)

Let (M, d) be a compact metric space, $\varphi : R \times M \to M$ be a continuous flow, and $\Lambda \subset M$ be a compact φ-invariant set. We say that the flow φ is Komuro-expansive in Λ if for any $\epsilon > 0$ there exists $\delta > 0$ so that if $x, y \in \Lambda$ and $d(\varphi_t(x), \varphi_{h(t)}(y)) < \delta$ for every $t \in \mathbb{R}$ and for some increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ then there is $t_0 \in \mathbb{R}$ such that $\varphi_{h(t0)}(y) \in \varphi_{[t0-\epsilon, t0+\epsilon]}(x)$.

We will say that φ is expansive when it is Komuro-expansive.

In the article "**Singular-hyperbolic attractors are chaotic**" of Zezé with Vitor Araujo, Enrique Pujals and Marcelo Viana, they proved, among other important results, that a singular-hyperbolic attractor of a 3-dimensional flow is expansive: if two points remain close for all times, possibly with time reparametrization, then their orbits coincide.

Theorem (Araujo, Pacifico, Pujals, Viana, *T.A.M.S.* 361)

Let Λ be a singular-hyperbolic attractor of $X \in X^1(M)$. Then Λ is expansive.

In that paper the authors extended to the class of singular-hyperbolic attractors the main elements of the ergodic theory of uniformly hyperbolic attractors for flows.

Entropy-expansiveness (*h*-expansiveness)

Zezé studied different variants of expansiveness, in particular entropy-expansiveness.

Entropy-expansiveness (h-expansiveness)

Zezé studied different variants of expansiveness, in particular entropy-expansiveness.

Definition

Let $f : M \to M$ be a homeomorphism and $K \subset M$ compact invariant. We say that f/K is **entropy-expansive** or h-**expansive** if and only if there exists $\epsilon > 0$ such that $\sup_{x \in K} h(f, \Gamma_\epsilon(x)) = 0$.

Here $h(f, \Gamma_\epsilon(x))$ is the topological entropy of f restricted to $\Gamma_\epsilon(x)$.

Entropy-expansiveness (h-expansiveness)

Zezé studied different variants of expansiveness, in particular entropy-expansiveness.

Definition

Let $f : M \to M$ be a homeomorphism and $K \subset M$ compact invariant. We say that f/K is **entropy-expansive** or h-**expansive** if and only if there exists $\epsilon > 0$ such that $\sup_{x \in K} h(f, \Gamma_\epsilon(x)) = 0$.

Here $h(f, \Gamma_\epsilon(x))$ is the topological entropy of f restricted to $\Gamma_\epsilon(x)$.

Entropy-expansiveness and domination.

Theorem (Pacifico, V., *Rev. Mat. Complutense* 21)

Let M be a compact boundaryless C^∞ surface and $f : M \to M$ be a C^r diffeomorphism such that $K \subset M$ is a compact f-invariant subset with a dominated splitting $E \oplus F$. Then f/K is h-expansive.

This theorem admits a partial converse:

Entropy-expansiveness and domination.

Theorem (Pacifico, V., *Rev. Mat. Complutense* 21)

Let M be a compact boundaryless C^∞ surface and $f: M \to M$ be a C^r diffeomorphism such that $K \subset M$ is a compact f-invariant subset with a dominated splitting $E \oplus F$. Then f/K is h-expansive.

This theorem admits a partial converse:

Theorem (Pacifico, V., *Rev. Mat. Complutense* 21)

Let M be a compact boundaryless C^∞ surface and $f: M \to M$ be a C^r diffeomorphism. Let $H(p)$ be an f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that the continuation $H(p_g)$ of $H(p)$ is h-expansive. Then $H(p)$ has a dominated splitting.

Definition

Let M be a compact C^∞ manifold, $dim(M) = d$, and $f : M \to M$ a C^1 diffeomorphism, $r \geq 1$. Let $H(p)$ be a homoclinic class associated to the hyperbolic periodic point p. We say that $f/H(p)$ is C^1-robustly h-expansive if there is a C^1 neighborhood $\mathcal{U}$ of f, such that for $g \in \mathcal{U}$ the continuation $H(p_g)$ of $H(p)$ is h-expansive.

Theorem (Pacifico, V., *Nonlinearity* 23)

Let M, $f : M \to M$ and $H(p)$ be as above. Assume moreover that $f/H(p)$ is isolated. Then for g in $\mathcal{U}(f)$, $H(p_g)$ has a dominated splitting of the form $E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$ where E is contracting, G is expanding and all F_j are not hyperbolic and $\dim(F_j) = 1$. Moreover, in case that the index of periodic points in $H(p_g)$ are in a C^1 robust way equal to index(p) then for an open dense subset $\mathcal{V} \subset \mathcal{U}(f)$, $H(p_g)$ is hyperbolic.

Symbolic extensions.

A dynamical system (X, f) has a *symbolic extension* if there exists a subshift (Y, σ) and a continuous surjective map $\pi : Y \to X$ such that $\pi \circ \sigma = f \circ \pi$. The system (Y, σ) is called an *extension* of (X, f) and (X, f) is called a *factor* of (Y, σ).

Theorem (Díaz, Fisher, Pacifico, V., *Disc. Cont. Dyn. Syst.* 32)

Let f be a diffeomorphism and Λ be a compact f-invariant set admitting a dominated splitting $E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u$, where E^s is uniformly contracting, E^u is uniformly expanding, and all E_i are one-dimensional. Then $f|_\Lambda$ is entropy-expansive and as a corollary $f|_\Lambda$ has a principal symbolic extension.

Other forms of expansivity

Carlos Morales has introduced some concepts related to expansiveness.

Definition (Morales)

Given a positive integer N, the homeomorphism f is *N-expansive* if there is $\alpha > 0$ such that $\sharp(\Gamma_\alpha(x)) \leq N$ for all $x \in M$. Here $\sharp A$ stands for the cardinal of the set A. That is, at most N orbits *α-shadow* the orbit of x by f.

With respect to this definition we found that

Theorem (Artigue, Pacifico, V.)

If $f : M \to M$ is a 2-expansive homeomorphism defined on a compact surface and $\Omega(f) = M$ then f is expansive.

Theorem (Artigue, Pacifico, V.)

There are 2-expansive homeomorphisms of surfaces that are not expansive.

Theorem (Artigue, Pacifico, V.)

If $f : M \to M$ is a 2-expansive homeomorphism defined on a compact surface and $\Omega(f) = M$ then f is expansive.

Theorem (Artigue, Pacifico, V.)

There are 2-expansive homeomorphisms of surfaces that are not expansive.

Indeed Artigue in a subsequent paper has shown that there are homeomorphisms which are robustly N-expansive in the C^r-topology, $r \geq 2$. These examples need not be Anosov. They are axiom A with tangencies at wandering points.

cw-expansiveness for flows

With Arbieto and Cordeiro, Pacifico defined continuum-wise expansivity (*cw*-expansivity) for flows and proved the following

Theorem (Arbieto, Cordeiro, Pacifico)

Let M be a compact metric space and $\{X^t\}_{t\in\mathbb{R}}$ a flow on M. If X^t is cw-expansive and the topological dimension of M is greater than 1, then the topological entropy of X^t is positive.

N-expansiveness for flows

Pacifico with Artigue and Cordeiro defined N expansiveness for flows. They also defined Komuro-N-expansiveness for flows. We summarize the main results of their paper in the following

Theorem (Artigue, Cordeiro, Pacifico)

1. *If a flow is N-expansive, $N > 0$, then it is continuum-wise-expansive (cw-expansive).*
2. *There are examples of cw-expansive flows defined on a compact metric space, but not N-expansive for any natural number N.*
3. *A flow on a compact surface is Komuro N-expansive if and only if it is Komuro expansive. There are Komuro cw-expansive flows on compact surfaces which are not Komuro N-expansive.*

Measure expansiveness

Definition (Morales)

Let $f : X \to X$ be a homeomorphism defined on a compact metric space (X, d) and μ a non-atomic probability measure defined on X. We say that f is a μ-expansive homeomorphism if there is $\alpha > 0$ such that $\mu(\Gamma_\alpha(x)) = 0$ for all $x \in X$.

Definition

A diffeomorphism $f : M \to M$ exhibits a homoclinic tangency if there is a hyperbolic periodic orbit O whose invariant manifolds $W^s(O)$ and $W^u(O)$ have a non transverse intersection.

Let us denote by HT to the subset of $Diff^1(M)$ exhibiting a homoclinic tangency.

Theorem (Pacifico, V. *Proc. Amer. Math. Soc.* 143)

Let $f : M \to M$ be a C^1-diffeomorphism defined on a compact manifold M. There is a $\mathcal{G}$ residual subset of $\mathrm{Diff}^1(M)\backslash\overline{HT}$ such that for any Borel probability measure μ (invariant by f or not) absolutely continuous with respect to Lebesgue, we have that there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in M$. In particular f is μ-expansive.

Theorem (Pacifico, V. *Proc. Amer. Math. Soc.* 143)

Let $f : M \to M$ be a C^1-diffeomorphism defined on a compact surface M having a homoclinic tangency associated to a hyperbolic periodic orbit O. Then there is an arbitrarily small C^1-perturbation of f giving a diffeomorphism $F : M \to M$ which is not measure-expansive.

Expansive Measures Versus Lyapunov Exponents

Pacifico with Alma Armijo investigate the relation between expansive measures and Lyapunov exponents. They proved:

Theorem (Armijo, Pacifico, *Proc. Amer. Math. Soc.* 146)

Let $f \in Diff^1_{loc}(M)$ and $\mu \in \mathcal{M}_f \setminus \mathcal{A}(M)$ an ergodic probability measure such that all of its Lyapunov exponents are positive. Then f is positively μ-expansive.

Theorem (Armijo, Pacifico, *Proc. Amer. Math. Soc.* 146)

Let $f \in Diff^1_{loc}(M)$ and suppose that there is a C^1-open neighborhood U of f such that every $g \in U$ is positively μ-expansive for all $\mu \in \mathcal{M}(M) \setminus \mathcal{A}(M)$. Then f is expanding.

Expansiveness versus Lyapunov exponents

With Zezé we have defined Lyapunov exponents for expansive homeomorphisms (not differentiability assumed) proving that the upper Lyapunov exponent $\Lambda^+(x)$ is positive.

Theorem (Pacifico, V., *Proc. Edinb. Math. Soc.* 63)

There is a residual subset $\mathcal{R}^+$ of M such that for every $x \in \mathcal{R}^+$ we have that $\Lambda^+(x) > 0$.

Recall that for expansive homeomorphisms Fathi has proved that there is a metric d and numbers $k > 1$, $\varepsilon_0 > 0$ such that: $\forall x, y \in M, \max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \min\{kd(x, y), \varepsilon_0\}$.

Theorem (Pacifico, V., *Proc. Edinb. Math. Soc.* 63)

Let (M, d) be a Peano space, with d Fathi's metric, $f : M \to M$ expansive and $\gamma = \log(k)$, $k > 1$, the constant above. Then for all $x \in M$ it holds $\Lambda^+(x) \geq \gamma$ and $\lambda^+(x) \leq -\gamma$.

Set valued expansive maps

Pacifico studied, in a joint work with Welington Cordeiro, *cw*-expansive set-valued maps.
They define the concept of continuum wise expansivity for set-valued maps $F : X \multimap X$ and among other results prove

Theorem (Cordeiro, Pacifico, *Proc. AMS* 144)

Let $F : X \multimap X$ be an upper semicontinuous cw-expansive topological dynamical system with constant of expansivity equals to $\delta > 0$. If dimtop$(X) > 0$ then the topological entropy $h_{top}(X; F)$ is positive.

Set valued expansive maps

Pacifico studied, in a joint work with Welington Cordeiro, *cw*-expansive set-valued maps.
They define the concept of continuum wise expansivity for set-valued maps $F : X \multimap X$ and among other results prove

Theorem (Cordeiro, Pacifico, *Proc. AMS* 144)

Let $F : X \multimap X$ be an upper semicontinuous cw-expansive topological dynamical system with constant of expansivity equals to $\delta > 0$. If dimtop$(X) > 0$ then the topological entropy $h_{top}(X; F)$ is positive.

Theorem (Cordeiro, Pacifico, *Proc. AMS* 144)

Let $F : X \multimap X$ be a topological dynamical system. If F has the point-wise specification property then the dynamical system is topologically mixing.

Recently **Pacifico** proposed to study expansive set-valued maps. A previous definition of this concept given by Richard Williams is used.

Definition

Let (X, d) be a compact metric space and $F : X \multimap X$ an u.s.c. set valued map. An F-suborbit of x, also called F-sequences in this paper, is a set of the form

$$\{x_j \ : \ x_0 = x, x_{j+1} \in F(x_j) \text{ for each } j \in \mathbb{Z}\}.$$

Definition (Richard Williams)

A set valued map $F : X \multimap X$ is RW-expansive on X with expansive constant $\delta > 0$ if $x, y \in X$, $x \neq y$ implies that for each F-suborbit A of x and for each F-suborbit B of y, there exist $x_n \in A$, $y_n \in B$ such that $d(x_n, y_n) > \delta$.

And so?

In a work in progress **Pacifico** and collaborators have defined Lyapunov exponents for set valued maps and prove that when F is a set valued expansive map the upper Lyapunov exponent is positive, the topological entropy is positive and also give some examples of set valued expansive maps on surfaces.
Covid19 has delayed this work.

And so?

In a work in progress **Pacifico** and collaborators have defined Lyapunov exponents for set valued maps and prove that when F is a set valued expansive map the upper Lyapunov exponent is positive, the topological entropy is positive and also give some examples of set valued expansive maps on surfaces.
Covid19 has delayed this work.

And the story continues ....

Zezé in Salto (Uruguay) 2010 I

Zezé in Salto (Uruguay) 2010 II

¡Happy birthday Zezé!