Robust transitivity and domination f/endo displaying critics

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Setting

C^1-endomorphisms on compact boundaryless manifolds

Goal

To find necessary conditions and topological obstructions for the existence of robustly transitive maps.

Diffeomorphisms

- Mañé'82: RT $\Leftrightarrow$ Anosov. (dim. 2)
- Shub'71: RT partially hyperbolic, but not hyperbolic (dim. 4).
- Mañé'78: Derived from Anosov (dim. 3).
- Bonatti-Viana'00: RT with dominated splitting, but not partially hyperbolic (dim. 4).

DPU'99 - BDP'03

C^1-RT $\Rightarrow$ DOMINATED SPLITTING

Local diffeomorphisms

- Shub 60's: Expanding maps.
- Gromov'81: Infranilmanifolds are the only manifolds admitting expanding maps.
- Lizana-Pujals'13: Necessary and sufficient conditions are given for having RT local diffeomorphisms
  ▷ volume expanding is C^1 necessary;
  ▷ it is not necessary to have a "weak form of hyperbolicity" ([BDP'03] does not hold!).

Endomorphisms w/critics

- Which homotopy classes admit C^1-robustly transitive endomorphisms displaying critical points?
- Which manifolds support these kind of maps?
- Is it necessary some “weak form of hyperbolicity”?

Examples on surfaces

- Berger-Rovella'13, Iglesia-Lizana-Portela'16

$$|\mu| \leq 1 < |\lambda|$$

- Lizana-Ranter'16-21

$$1 < |\mu| < |\lambda|, \quad |\mu| = 0 < 1 < |\lambda|$$

▷ robustly transitive;
▷ persistent critical points;
▷ exhibit family of unstable cones;
▷ $dim(Ker(Df)) \leq m - 1$ (necessary condition);
▷ exhibit a weak form of hyperbolicity.

On surfaces

- Lizana-Ranter'21 [Adv. in Mathematics 390, 2021]

  ▷ Theorem A. f robustly transitive and $Cr(f) \neq \emptyset$
  $\Rightarrow$ f is partially hyperbolic.

  ▷ Theorem B. M admits RT endomorphism
  $\Rightarrow$ M is either $\mathbb{T}^2$ or $\mathbb{K}^2$.

  ▷ Corollary. There are no robustly transitive on $\mathbb{S}^2$.

On surfaces

- Lizana-Ranter'21

▷ Theorem C. f transitive with dominated splitting
$\Rightarrow$ f is homotopic to a linear map having at least one eigenvalue with modulus larger than one.

▷ Corollary. f is robustly transitive
$\Rightarrow$ f is homotopic to a linear map having at least one eigenvalue with modulus larger than one.

▷ Corollary. There are no robustly transitive surface endomorphisms homotopic to the identity.

On higher dimension

- Lizana-Potrie-Pujals-Ranter'21

  ▷ Theorem A. Every robustly transitive endomorphism displaying critical points admits a nontrivial dominated splitting.

  ▷ Corollary. Even dimensional sphere do not admit robustly transitive endomorphisms.

Sketch of the Proof

- f_0 RT displaying critical points and $\mathcal{U}_0(f_0)$.
- (Key Lemma) There are no robustly transitive endomorphisms exhibiting full-dimensional kernel.
- There exists $1 \leq \kappa < d$ the smallest integer satisfying

$$\dim \ker(Df^m) \leq \kappa, \forall f \in \mathcal{U}_0, \ m \geq 1, \tag{1}$$

where $\dim \ker(Df) = \max_{x \in M} \dim \ker(Df_x)$.

Sketch of the Proof

- f_0 can be approximated by f satisfying equality in (1). Let m_f the smallest positive integer such that
  - $\{x \in M : \dim\ker(Df^m_x) = \kappa\}$ has nonempty interior for some $x \in M$; or
  - if such subset above has empty interior then we take m_f as the smallest one such that $\dim\ker(Df^m_x) = \kappa$, for some $x \in M$.

- $\mathrm{Cr}_\kappa(f) := \{x \in M : \dim\ker(Df^{m_f}_x) = \kappa\}$.

- $\mathcal{F}_0$ - the set of all endomorphisms f in $\mathcal{U}_0$ which $\mathrm{Cr}_\kappa(f)$ has nonempty interior.

Sketch of the Proof

- For $f \in \mathcal{F}_0$

$$\Lambda_f = \left\{ (x_i)_i \subseteq M \;\middle|\; \begin{array}{l} x_{i+1} = f(x_i),\ \forall i \in \mathbb{Z}, \exists (i_n)_n \subseteq \mathbb{Z} \text{ s.t.} \\ x_{i_n} \in \mathrm{Cr}_\kappa(f) \text{ for infinitely many } i_n \end{array} \right\}$$

- $E \oplus F$ candidate for the invariant (dominated) splitting over Λ_f.
- The splitting is extended to the closure of Λ_f.