Self consistent transfer operators in a weak and not so weak coupling regime. Invariant measures, convergence to equilibrium, linear response.

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What kind of systems we are thinking about

- Consider a complex system in which you have many interacting elementary subsystems each of them following the same dynamics plus some perturbation coming from the collective state of the other subsystems, for instance:
  - A set of particles in the same environment, following the dynamics of an external field plus the mutual interactions.
  - A set of agents following a certain evolution law depending on some internal dynamics plus a perturbation coming from the collective state of the other agents.
  - (The perturbation will be a function of the distribution of the states of the many systems in the (common) phase space).
- The behavior of this kind of systems can be studied by *self consistent transfer operators*.

Transfer operator associated to a dynamical system

- Many important results on the statistical properties of dynamics have been achieved by transfer operator methods.
- The transfer operator associated to a dynamical system shows how the dynamics moves measures in the phase space.
- This is a linear operator between spaces of measures with sign or suitable distributions spaces.
- The self consistent operators play the same role, they are Markov in a certain sense but are *nonlinear*.

Transfer operator associated to a dynamical system

- For Linear transfer operators there are many tools to answer the basic questions:
  - is there a regular, physical invariant measure for the system?
  - Is this unique?
  - Will there be convergence to equilibrium?
  - Is the "physical" measure stable under changes in the system?

The context

- For self consistent transfer operators these questions have been investigated in classes of examples, showing a rich and complicated behavior, using different approaches.
- Mathematical results in this context have been obtained in classes of examples mostly related to globally coupled maps by different approaches (Balint, Bardet, Blank, Chazottes, Fernandez, Gottwald, Keller, Selley, Toth, Tanzi, Wormell, Zweimuller and others.)

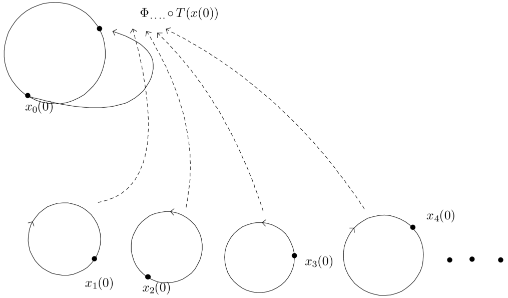
Overview

- In this work I try to approach these questions from a general point of view, trying to understand what kind of assumptions are sufficient to give a general (existence, uniqueness, convergence, stability, response) result.
- We show a general approach to the subject by the use of strong-weak spaces similar to what is succesfully done in many cases of deterministic or random dynamical systems.

The background

Coupled maps

- Consider a set of identical maps $(\mathbb{S}^1, T)_i$, $i \in M$ in which the dynamics of each site is perturbed by the average perturbation coming from the state of the other sites.
- Orbit $x_i(t)$ for a given initial condition $x_i(0)$.
- $\mathbf{x}(t) := (x_i(t)) \in [\mathbb{S}^1]^M$: **global state** of the dynamics at time t



Coup. maps, more precisely

- For simplicity consider the unit circle $\mathbb{S}^1$ as a phase space and we will equip $\mathbb{S}^1$ with the Borel $\sigma-$algebra.
- Let us consider an additional metric space M equipped with the Borel $\sigma-$algebra and a probability measure $p \in PM(M)$.
- Let us consider a collection of *identical dynamical systems* $(\mathbb{S}^1, T)_i$, with $i \in M$ and $T : \mathbb{S}^1 \to \mathbb{S}^1$ being a Borel measurable function.
- Initial state $\mathbf{x}(0) = (x_i(0))_{i\in M} \in (\mathbb{S}^1)^M$ (we suppose $i \to x_i(0)$ being measurable).

Coup. maps, the dynamics

We now define the dynamics $\mathcal{T} : (\mathbb{S}^1)^M \to (\mathbb{S}^1)^M$ by

$$\mathbf{x}(t+1) := \mathcal{T}(\mathbf{x}(t))$$

$\mathbf{x}(t+1)$ is defined on every coordinate by applying at each step the local dynamics T, *plus a perturbation given by the mean field interaction with the other systems*

$$x_i(t+1) = \Phi_{\delta,\mathbf{x}(t)} \circ T(x_i(t)) \tag{1}$$

for each $i \in M$.

- Here $\Phi_{\delta,\mathbf{x}(t)} : \mathbb{S}^1 \to \mathbb{S}^1$ represents the perturbation provided by the global mean field coupling with strength $\delta \geq 0$.

Coup. maps, the dynamics

$\Phi_{\delta,\mathbf{x}(t)} : \mathbb{S}^1 \to \mathbb{S}^1$ represents the perturbation provided by the global mean field coupling with strength $\delta \geq 0$

- consider a cont. $h : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ and $h(x, y)$ represents the way in which the presence of some system in the state y perturbs the systems in the state x.
- Def. $\Phi_{\delta,\mathbf{x}(t)} : \mathbb{S}^1 \to \mathbb{S}^1$ as

$$\Phi_{\delta,\mathbf{x}(t)}(x) = x + \delta \int_M h(x, x_j(t))\, dp(j).$$

We remark that with this definition, for each $t \in \mathbb{N}$, $i \to x_i(t)$ is also measurable.

Coup. maps, the dynamics

- We say that the global state $\mathbf{x}(t)$ of the system is represented by $\mu_{\mathbf{x}(t)} \in PM(\mathbb{S}^1)$ if

$$\mu_{\mathbf{x}(t)} = (I_{\mathbf{x}(t)})_*(p)$$

where $I_{\mathbf{x}(t)} : M \to \mathbb{S}^1$, is defined by $I_{\mathbf{x}(t)}(j) = x_j(t)$.

Fact

Consider $((\mathbb{S}^1)^M, \mathcal{T})$ *as above. Let* $\mu \in PM(\mathbb{S}^1)$, *consider*

$$\Phi_{\delta,\mu}(x) := x + \delta \int_{\mathbb{S}^1} h(x,y)\ d\mu(y).$$

Suppose $\mathbf{x}(0)$ *is represented by a measure* $\mu_{\mathbf{x}(0)}$, *then* $\mathbf{x}(1) = \mathcal{T}(\mathbf{x}(0))$ *is represented by*

$$\mu_{\mathbf{x}(1)} = L_{\Phi_{\delta,\mu_{\mathbf{x}(0)}} \circ T}(\mu_{\mathbf{x}(0)}).$$

(where L_F is the transf. op. associated to a map F)

SeCoTrOp, formally, notations

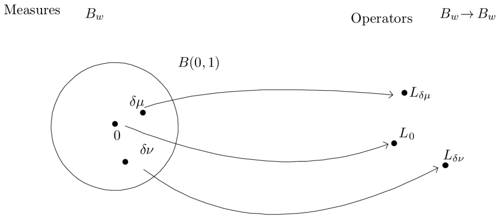
- Let X be a metric space and $SM(X)$ be the set of signed Borel measures on X.
- Let $B_w \subseteq SM(X)$ be a normed vector subspace of $SM(X)$
- Let $P_w \subseteq B_w$ the set of probability measures

SeCoTrOp, formally, notations

- A self consistent transfer operator will be the given of a family of Markov linear operators such that $L_{\delta,\mu} : B_w \to B_w$ for each $\mu \in P_w$, some $\delta \geq 0$ and the dynamical system $(P_w, \mathcal{L}_\delta)$ where $\mathcal{L}_\delta : P_w \to P_w$ is defined by

$$\mathcal{L}_\delta(\mu) = L_{\delta,\mu}(\mu).$$

- Typical situation when $L_{\delta,\mu} = L_{\delta\mu}$



SeCoTrOp, standing assumptions

If A, B are two normed vector spaces and $L : A \to B$ is a linear operator we denote the mixed norm $\|L\|_{A\to B}$ as

$$\|L\|_{A\to B} := \sup_{f\in A, \|f\|_A \leq 1} \|Lf\|_B.$$

- Let B_s be a normed vector space s.t. $(B_s, || \, ||_s) \subseteq (B_w, || \, ||_w) \subseteq SM(X)$; $|| \, ||_s \geq || \, ||_w$.
- Let us suppose that the linear form $\mu \to \mu(X)$ is continuous on B_w.
- Let $P_w :=$ probability measures in B_w and $P_s := P_w \cap B_s$. Suppose P_w is complete.
- Let us fix $\delta \geq 0$, suppose that when μ varies in P_w the family $L_{\delta,\mu}$ is such that $||L_{\delta,\mu}||_{B_s\to B_s} \leq M$, $||L_{\delta,\mu}||_{B_w\to B_w} \leq M$
- Suppose $L_{0,\mu_1} = L_{0,\mu_2} := L_0$ for each $\mu_1, \mu_2 \in P_w$.

Skew product associated and the importance of inv. meas.

In the case where $L_{\delta,\mu}$ is the transfer operator associated to $T_{\delta,\mu} : \mathbb{S}^1 \to \mathbb{S}^1$ one can consider $(B_w \times \mathbb{S}^1, F)$ where $F : B_w \times \mathbb{S}^1 \to B_w \times \mathbb{S}^1$ is defined by

$$F(\mu, x) = (\mathcal{L}_\delta(\mu), T_{\delta,\mu}(x)).$$

- If μ is a fixed point for $\mathcal{L}_\delta$ the dynamics is nontrivial only on the second coordinate, where $T_{\delta,\mu}$ is a map for which μ is invariant.
- Finding the fixed points of $\mathcal{L}_\delta$ gives important information on the statistical behavior of the second coordinate of the system.

SeCoTrOp, existence of the invariant measure

Theorem (Exist. part 1)

Suppose there is $\pi_n : B_w \to B_s$, linear Markov projection of rank n satisfying: there is $M \geq 0$ and a decreasing $a(n) \to 0$ s. t. for each $n \geq 0$

$$\begin{aligned} ||\pi_n||_{B_w \to B_w} &< M, \\ ||\pi_n||_{B_s \to B_s} &< M \end{aligned} \quad (2)$$

and

$$||\pi_n f - f||_w \leq a(n)||f||_s. \quad (3)$$

Suppose $\pi_n(P_w) \subseteq P_s$ and $\pi_n(P_w)$ is bounded in B_s. Let us fix $\delta \geq 0$ and suppose that:...

Theorem (Exist. part 2)

...

Exi1 *there is* $M_1 \geq 0$ *such that* $\forall \mu_1 \in P_w$ *and* $f \in P_w$ *which is a fixed point of* L_{δ,μ_1} *it holds*

$$||f||_s \leq M_1;$$

Exi1.b $\forall \mu_1 \in P_w,\ n \in \mathbb{N}$ *and for each* $f \in P_w$ *which is a fixed point for the finite rank approximation* $\pi_n L_{\delta,\pi_n \mu_1} \pi_n$ *of* L_{δ,μ_1} *it holds*

$$||f||_s \leq M_1;$$

Exi2 *there is* $K_1 \geq 0$ *such that* $\forall \mu_1, \mu_2 \in P_w$

$$||L_{\delta,\mu_1} - L_{\delta,\mu_2}||_{B_s \to B_w} \leq \delta K_1 ||\mu_1 - \mu_2||_w.$$

Then there is $\mu \in P_s$ *such that* $\mathcal{L}_\delta \mu = \mu$ *and* $||\mu||_s \leq M_1$.

- **Idea of the proof**: for each n use the Brouwer fixed point theorem to find a fixed point $\mu_n \in P_w$ for $\mu \to \pi_n L_{\delta,\pi_n\mu}\pi_n\mu$. The assumptions implies that μ_n has a subsequence converging in the weak norm $\mu_n \to \mu$ because of the completeness of P_w. By the other regularity assumptions this will be a fixed point for $\mu \to L_{\delta,\mu}\mu$.
- The theorem applies to all to all coupled expanding maps, coupled systems with additive noise and other. Even if δ is relatively large. (δ preserving the expansiveness in the first case)

Corollary

Let us consider a coupled maps system as before with $T_0 \in C^0(\mathbb{S}^1 \to \mathbb{S}^1)$, $h \in Lip(\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R})$ *and* $\delta \geq 0$, *then there is* $\mu \in PM(\mathbb{S}^1)$ *such that*

$$\mathcal{L}_\delta(\mu) = \mu.$$

- **Idea of the proof**: We apply Theorem 2 with $|| \ ||_w$, $|| \ ||_s$ defined by $||\mu||_w = \sup_{g \in Lip(\mathbb{S}^1 \to \mathbb{R}), ||g||_{Lip} \leq 1} \int g \ d\mu$ and $||\mu||_s = \mu^+(\mathbb{S}^1) + \mu^-(\mathbb{S}^1)$ (the total variation norm). π_n is a suitable projection on piecewise linear densities on a grid.

Existence and uniqueness of the invariant measure

Theorem (Uniq.+ approx)

Suppose $L_{\delta,\mu}$ *satisfies* $(Exi1)$ *and* $(Exi2)$*. Suppose* $\exists \mu \in P_w$ *with* $||\mu||_w \leq M_1$*. Suppose* $\exists \overline{\delta} > 0$ *such that for each* $0 \leq \delta < \overline{\delta}$ *and* $\mu \in P_w$ *with* $||\mu||_w \leq M_1$*,* $L_{\delta,\mu}$ *has a unique fixed probability measure in* P_w *we denote by* f_μ*. Suppose:*

- Exi3 *there is* $K_2 \geq 1$ *such that* $\forall \mu_1, \mu_2 \in P_w$ *with* $\max(||\mu_1||_w, ||\mu_2||_w) \leq M_1$

$$||f_{\mu_1} - f_{\mu_2}||_w \leq \delta K_2 ||\mu_1 - \mu_2||_w.$$

Then for each $0 \leq \delta \leq \min(\overline{\delta}, \frac{1}{K_2})$*, there is a unique* $\mu \in P_w$ *such that*

$$\mathcal{L}_\delta(\mu) = \mu.$$

Furthermore $\mu = \lim_{k\to\infty} \mu_k$ *where* $\mu_0 \in P_w$ *with* $||\mu_0||_w \leq M_1$*, then* μ_i *is the fixed probability measure of* $L_{\delta,\mu_{i-1}}$ *in* P_w *and so on.*

- **Idea of the proof**: By *Exi*3 μ_k is a Cauchy sequence converging geometrically in P_w,since this is complete it must converge to μ. By the other regularity assumptions we get that μ is a fixed point for $\mu \to L_{\delta,\mu}\mu$.
- The theorem applies to all to all coupled expanding maps, coupled systems with additive noise and other when δ is small.

Exponential convergence to equilibrium

- Let us consider a sequence of normed vector spaces $(B_{ss}, || \ ||_{ss}) \subseteq (B_s, || \ ||_s) \subseteq (B_w, || \ ||_w) \subseteq SM(X)$ with norms satisfying $|| \ ||_{ss} \geq || \ ||_s \geq || \ ||_w$.
- Let us consider a parameter $0 \leq \delta < 1$ and family of Markov bounded operators $L_{\delta,\mu} : B_i \rightarrow B_i$ satisfying the standing assumptions.

Suppose furthermore that the family $L_{\delta\mu}$ satisfies the following...

Exponential convergence to equilibrium

Con1 $L_{\delta,\mu}$ satisfy a common "one step" Lasota Yorke inequality. $\exists$ $B, \lambda_1 \in \mathbb{R}$ with $\lambda_1 < 1$ such that $\forall f \in B_s$, $\mu \in P_w$

$$||L_{\delta,\mu} f||_w \leq ||f||_w \quad (4)$$

$$||L_{\delta,\mu} f||_s \leq \lambda_1 ||f||_s + B||f||_w . \quad (5)$$

Con2 (extended $(Exi2)$ property): there is $K \geq 1$ such that $\forall f \in B_s$, $\mu, \nu \in B_w$

$$||(L_{\delta,\mu} - L_{\delta,\nu})(f)||_w \leq \delta K ||\mu - \nu||_w ||f||_s$$

$$||(L_0 - L_{\delta,\nu})(f)||_w \leq \delta K ||\nu||_w ||f||_s$$

and $\forall f \in B_{ss}$, $\mu, \nu \in B_w$

$$||(L_{\delta,\mu} - L_{\delta,\nu})(f)||_s \leq \delta K ||\mu - \nu||_w ||f||_{ss} .$$

...

Exponential convergence to equilibrium

...

Con3 L_0 has convergence to equilibrium: $\exists\ a_n \geq 0,\ a_n \longrightarrow 0$ s. t., setting

$$V_s = \{\mu \in B_s | \mu(X) = 0\}$$

we get

$$\forall v \in V_s,\ ||L_0^n(v)||_w \leq a_n ||v||_s. \tag{6}$$

Exponential convergence to equilibrium

Theorem Let $L_{\delta,\mu}$ be a family of Markov operators $L_{\delta,\mu} : B_i \longrightarrow B_i$ and $\delta \leq 1$. Suppose $L_{\delta\mu}$ satisfy $(Con1)$, ..., $(Con3)$ and

$$\sup_{\mu \in B_w(0,1), \delta \leq \hat{\delta}} ||L_{\delta\mu}||_{B_{ss} \to B_{ss}} < +\infty. \tag{7}$$

Let

$$\mathcal{L}_\delta(\mu) = L_{\delta\mu}\mu.$$

Suppose that $\forall \delta \leq \hat{\delta}$ there is invariant $\mu_\delta \in P_w$ for $\mathcal{L}_\delta$ and

$$\lim_{\delta \to 0} ||\mu_\delta||_{ss} < +\infty. \tag{8}$$

Then $\exists \ \overline{\delta}$ s. t. $0 < \overline{\delta} < \hat{\delta}$ and $C, \gamma \geq 0$ s.t. $\forall \ 0 < \delta < \overline{\delta}$, $\nu \in B_{ss}$, $\nu \in P_w$ we have

$$||\mathcal{L}_\delta^n(\nu) - \mu_\delta||_s \leq Ce^{-\gamma n}||\nu - \mu_\delta||_s.$$

- **Idea of proof.** We need to estimate $||\mathcal{L}_\delta^n(\nu) - \mu_\delta||_s$. Denote $\nu_1 = \nu$ and $\nu_n = L_{\delta,\nu_{v-1}}\nu_{n-1}$. $\mathcal{L}_\delta^n(\nu)$ can be seen as a sequential composition

$$\mathcal{L}_\delta^n(\nu) = L(n)(\nu)$$

$$L(n) = L_1 \circ L_2 \circ ... \circ L_n.$$

and $L_i = L_{\delta,\nu_i}$. Let us estimate this by

$$||L(n)(\nu) - \mathcal{L}_\delta^n(\mu_\delta)||_s \leq ||L(n)(\nu) - L(n)(\mu_\delta)||_s + ||L(n)(\mu_\delta) - \mathcal{L}_\delta^n(\mu_\delta)$$

- This also applies to coupled expanding maps, coupled systems with additive noise and other systems.

What is Linear response, informally?

- Initial system and invariant measure: (F_0, μ_0)
- Suppose after perturbation $\delta \dot{F}$ of "direction" $\dot{F}$ and strenght $\delta \in [0, \overline{\delta})$ we get a family (F_δ, μ_δ)
- The linear response of the invariant measure of the system with respect to the perturbations is represented by

$$\lim_{\delta \to 0} \frac{\mu_\delta - \mu_0}{\delta} = \dot{\mu}$$

converging in some stronger or weaker sense depending on the system or the perturbation. (L^p,distributions, etc...)

- We then get this first order development

$$\mu_\delta = \mu_0 + \dot{\mu}\delta + o(\delta). \tag{9}$$

Linear Response

- Let X be a compact metric space, $(B_{ss}, \| \|_{ss}) \subseteq (B_s, \| \|_s) \subseteq (B_w, \| \|_w) \subseteq SM(X)$ satisfying
$$\| \|_w \leq \| \|_s \leq \| \|_{ss}.$$
- the linear form $\mu \to \mu(X)$ is continuous on B_i, for $i \in \{ss, s, w\}$.
- Consider $V_{ss} \subseteq V_s \subseteq V_w$ of zero average measures defined as:
$$V_i := \{\mu \in B_i | \mu(X) = 0\}$$
where $i \in \{ss, s, w\}$.
- Consider $\mathcal{L}_\delta : B_i \to B_i$, s. t. $\mathcal{L}_\delta$ preserves positive measures, $\mathcal{L}_\delta(0) = 0$ and such that for each $\mu \in SM(X)$ it holds $[\mathcal{L}_\delta(\mu)](X) = \mu(X)$. We call such a family, a family of *"nonlinear" Markov operators.*

Theorem ((stat stability))

Let $\mathcal{L}_\delta : B_s \to B_s$ $\mathcal{L}_\delta : B_{ss} \to B_{ss}$ *with* $\delta \in \left[0, \overline{\delta}\right)$ *be "nonlinear" Markov op. Suppose* $\mathcal{L}_0$ *is linear bounded on* B_s *and* $\forall\ \delta \in \left[0, \overline{\delta}\right)$ $\exists$ *a prob. meas.* $h_\delta \in B_{ss}$ *s. t.* $\mathcal{L}_\delta h_\delta = h_\delta$. *Suppose:*

(SS1) *(regularity bounds)* $\exists\ M \geq 0$ *s. t.* $\forall\ \delta \in \left[0, \overline{\delta}\right)$

$$\|h_\delta\|_{ss} \leq M.$$

(SS2) *(convergence to equilibrium)* $\exists\ a_n \to 0$ *s. t.* $\forall\ g \in V_{ss}$

$$\|\mathcal{L}_0^n g\|_s \leq a_n ||g||_{ss};$$

(SS3) *(small perturbation) Let* $B_{2M} = \{x \in B_{ss}, ||x||_{ss} \leq 2M\}$. $\exists\ K \geq 0$ *s.t.* $\mathcal{L}_0 - \mathcal{L}_\delta : B_{2M} \to B_s$ *is* $K\delta$*-Lipschitz.*

Then

$$\lim_{\delta \to 0} \|h_\delta - h_0\|_s = 0.$$

Theorem (Linear Response)

Let $\mathcal{L}_\delta : B_s \to B_s$ $\mathcal{L}_\delta : B_{ss} \to B_{ss}$ *with* $\delta \in [0, \overline{\delta})$ *as before. Suppose* $\mathcal{L}_0$ *is linear and* $(SS1), (SS2), (SS3)$ *are satisfied. Suppose*

(LR1) *(resolvent of the unperturbed operator)* $(Id - \mathcal{L}_0)^{-1} := \sum_{i=0}^{\infty} \mathcal{L}_0^i$ *is bounded* $V_w \to V_w$.

(LR2) *(small perturbation and derivative operator) Let* $\overline{B}_{2M} = \{x \in B_s, ||x||_s \leq 2M\}$. $\exists$ $K \geq 0$ *s. t.* $\mathcal{L}_0 - \mathcal{L}_\delta : \overline{B}_{2M} \to B_w$ *is* $K\delta$*-Lipschitz.* $\exists$ $\dot{\mathcal{L}}h_0 \in V_w$ *s. t.*

$$\lim_{\delta \to 0} \left\| \frac{(\mathcal{L}_\delta - \mathcal{L}_0)}{\delta} h_0 - \dot{\mathcal{L}}h_0 \right\|_w = 0. \tag{10}$$

Then

$$\lim_{\delta \to 0} \left\| \frac{h_\delta - h_0}{\delta} - (Id - \mathcal{L}_0)^{-1} \dot{\mathcal{L}}h_0 \right\|_w = 0. \tag{11}$$

Examples

The above results can be applied to

- coupled expanding maps
- maps with additive noise
- other examples
- coupling different maps

Examples, coupling different maps

- Let us consider two different C^6 expanding maps of the circle $(T_1, S^1), (T_2, S^1)$.
- Given two probability densities $\psi_1, \psi_2 \in L^1(\mathbb{S}^1, \mathbb{R})$ representing the distribution of probability of the states in the two systems, two coupling functions $h_1, h_2 \in C^6(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R})$ and $\delta \in [-\epsilon_0, \epsilon_0]$ representing the way in which these distributions perturb the dynamics (which can be different for the two different systems)
- Let $\pi : \mathbb{S}^1 \to \mathbb{R}$ be the natural, universal covering projection, let us define $\Phi_{\delta,\psi_1,\psi_2} : \mathbb{S}^1 \to \mathbb{S}^1$ with $i \in \{1, 2\}$ as

$$\Phi_{\delta,\psi_i,\psi_2}(x) = x + \pi(\delta \int_{\mathbb{S}^1} h_1(x, y)\psi_1(y)dy + \delta \int_{\mathbb{S}^1} h_2(x, y)\psi_2(y)dy)$$

Examples, coupling different maps

- Denote by Q_{δ,ψ_i,ψ_2} the transfer operator associated to $\Phi_{\delta,\psi_i,\psi_2}$,defined as
$$[Q_{\delta,\psi_i,\psi_2}(\phi)](x) = \frac{\phi(\Phi^{-1}_{\delta,\psi_i,\psi_2}(x))}{|\Phi'_{\delta,\delta,\psi_i,\psi_2}(\Phi^{-1}_{\delta,\delta,\psi_i,\psi_2}(x))|}$$
for any $\phi \in L^1(\mathbb{S}^1, \mathbb{R})$.
- Now we consider a global system $(\mathbb{S}^1 \times \mathbb{S}^1, F_{\delta,\psi_i,\psi_2})$ with
$$F_{\delta,\psi_i,\psi_2}(x_1, x_2) = (\Phi_{\delta,\psi_i,\psi_2} \circ T_1(x_1), \Phi_{\delta,\psi_i,\psi_2} \circ T_2(x_2)).$$

Examples, coupling different maps

- Finally let us consider the space of functions $B_1 := \{(f_1, f_2) \in L^1(\mathbb{S}^1) \times L^1(\mathbb{S}^1)\}$ with the norm $||(f_1, f_2)||_{B_1} = ||f_1||_{L^1} + ||f_2||_{L^1}$ (this is also called the direct sum $L^1(\mathbb{S}^1) \oplus L^1(\mathbb{S}^1)$)
- and the stronger spaces $B_2 := \{(f_1, f_2) \in W^{1,1}(S^1) \times W^{1,1}(S^1)\}$ with the norm $||(f_1, f_2)||_{B_1} = ||f_1||_{W^{1,1}} + ||f_2||_{W^{1,1}}$ and $B_3 := \{(f_1, f_2) \in W^{2,1}(S^1) \times W^{2,1}(S^1)\}$ with the norm $||(f_1, f_2)||_{B_1} = ||f_1||_{W^{2,1}} + ||f_2||_{W^{2,1}}$ (again direct sums of Sobolev spaces).
- We define a family of transfer operators $L_{\delta,\phi_1,\phi_2} : B_w \to B_w$ depending on elements of the weaker space B_1 as

$$L_{\delta,\phi_1,\phi_2}((f_1, f_2)) = (Q_{\delta,\phi_1,\phi_2}(L_{T_1}(f_1)), Q_{\delta,\phi_1,\phi_2}(L_{T_2}(f_2))). \tag{12}$$

Examples, coupling different maps

- By this we can define the self consistent transfer operator $\mathcal{L}_\delta : B_w \to B_w$ associated to this system as

$$\mathcal{L}_\delta((f_1, f_2)) = L_{\delta, f_1, f_2}((f_1, f_2)). \tag{13}$$

We remark that B_1 can be identified with a closed subset of $L^1(\mathbb{S}^1 \times \mathbb{S}^1)$ by $(f_1, f_2) \to f$ where f is defined by $f(x, y) = f_1(x) f_2(y)$ and $\mathcal{L}_\delta$ preserves this subspace.

Examples, coupling different maps

Theorem

Let T_1, T_2 *be two* C^6 *expanding maps and let* $h_1, h_2 \in C^6(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R})$*. Let us consider a globally coupled system as defined above. There is some* $\overline{\delta}$ *such that for each* $\delta \in [0, \overline{\delta}]$ *there is* $(f_{1,\delta}, f_{2,\delta}) \in B_2$ *such that*

$$\mathcal{L}_\delta((f_{1,\delta}, f_{2,\delta})) = (f_{1,\delta}, f_{2,\delta}).$$

Furthermore there is $M \geq 0$ *such that for each* $\delta \in [0, \overline{\delta}]$

$$||(f_{1,\delta}, f_{2,\delta})||_{B_2} \leq M.$$

The control of the statistical properties

- Suppose we have F_0 and a set P of "*allowed infinitesimal perturbations*" we can apply to the system.
  (each $p \in P$ correspond to a suitable first order "germ" of some family F_δ, μ_δ)
- Suppose that with a perturbation $p \in P$ we know that

$$\frac{\mu_\delta - \mu_0}{\delta} \rightarrow \dot{\mu}$$

- Denote the linear response as $R(p) = \dot{\mu}$, $R : P \rightarrow \mathcal{M}$
- Suppose we want to influence a system in a way to *"push" its statistical properties towards a wanted direction*. .
- Given $g \in L^1$ which $p \in P$ minimizes or maximizes $\int g \; dR(p) = \frac{d(time\ \ avg.(g))}{d\delta}$?
- Existence and uniqueness results when P is convex.

Thanks and Parabens Zeze!

(the results presented are on Arxiv, to be updated soon)

Why this model (coupled maps)

- We define a global map $\mathcal{T} : [\mathbb{S}^1]^{\mathbb{N}} \to [\mathbb{S}^1]^{\mathbb{N}}$, where $x(t+1) := \mathcal{T}(x(t))$ and $x(t+1)$ is defined coordinatewise by

$$x_i(t+1) = (\Phi_{\delta,x(t)} \circ T(x_i(t)))_{i \in \mathbb{N}} \tag{14}$$

- $\Phi_{\delta,x(t)} : \mathbb{S}^1 \to \mathbb{S}^1$ represents the perturbation provided by the global coupling with strength δ, defined by

$$\Phi_{\delta,x(t)}(x) = x + \delta \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h(x, x_j(t))$$

  for some continuous $h : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$. $h(x,y)$
- (small perturbation of the identity).
- The extended system we consider is hence identified by the choice of $\mathbb{S}^1$, T, δ and h and can be denoted by $(\mathbb{S}^1, T, \delta, h)$.

Why this model (coupled maps and tran. op.)

- We say that the state $x(t)$ is distributed on $\mathbb{S}^1$ according to some measure μ if
$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j(t)} = \mu \tag{15}$$
in the weak topology.
- Then
$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h(x, x_j(t)) = \int h(x,y) d\mu(y).$$

Why this model (coupled maps and tran. op.)

- Let $k>1$ and $T_0 \in C^k(\mathbb{S}^1,\mathbb{S}^1)$ be a nonsingular map. Let us denote the transfer operator associated to T_0 by $L_{T_0} : L^1(\mathbb{S}^1,\mathbb{R}) \to L^1(\mathbb{S}^1,\mathbb{R})$.
In this case, given any density $\phi \in L^1(\mathbb{S}^1,\mathbb{R})$ the action of the operator on the density can then be described by the explicit formula

$$[L_0(\phi)](x) := \sum_{y \in T^{-1}(x)} \frac{\phi(y)}{|T'(y)|}.$$

Why this model (coupled maps and tran. op.)

- Given $h \in C^k(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R})$, $\delta \in [-\epsilon_0, \epsilon_0]$ and (a probability density) $\psi \in L^1(\mathbb{S}^1, \mathbb{R})$, let $\pi : \mathbb{S}^1 \to \mathbb{R}$ be the universal covering projection, define $\Phi_{\delta,\psi} : \mathbb{S}^1 \to \mathbb{S}^1$ as

$$\Phi_{\delta,\psi}(x) = x + \pi(\delta \int_{\mathbb{S}^1} h(x,y)\psi(y)dy).$$

- We assume ϵ_0 is so small so that $\Phi_{\delta,\psi}$ is a diffeomorphism for each $\delta \in [-\epsilon_0, \epsilon_0]$ and $\Phi'_{\delta,\psi} > 0$. Denote by $Q_{\delta,\psi}$ the transfer operator associated to $\Phi_{\delta,\psi}$,defined as

$$[Q_{\delta,\psi}(\phi)](x) = \frac{\phi(\Phi^{-1}_{\delta,\psi}(x))}{|\Phi'_{\delta,\psi}(\Phi^{-1}_{\delta,\psi}(x))|} \tag{16}$$

for any $\phi \in L^1(S^1, \mathbb{R})$ (we remark that $Q_{\delta,\psi}$ depends on the product $\delta\psi$).

$$\Phi_{\delta,\psi}(x) = x + \pi(\delta \int_{\mathbb{S}^1} h(x,y)\psi(y)dy).$$

Small perturbations of ψ in L^1 gives small perturbations on $\Phi_{\delta,\psi}$ in smoother norms depending on how smooth is h.

Why this model (coupled maps and tran. op.)

- We can hence consider a family of transfer operators $L_{\delta,\psi} : L^1 \longrightarrow L^1$ depending on ψ and δ as

$$L_{\delta,\psi} := Q_{\delta,\psi} \circ L_0.$$

- The operator $L_{\delta,\psi}$ can be seen as the transfer operator associated to the dynamics of a given node of the network of coupled systems, given that the distribution of probability of the states of the other nodes in the network is represented by ψ.

Why this model (coupled maps and tran. op.)

- Now suppose the initial conditions of the system $x(0)$ is distributed according to a measure having density ϕ
- after one iterate of the dynamics, $x(1)$ will be distributed as $L_{\delta,\phi}\phi$.
- This lead to the definition the nonlinear *self consistent transfer operator* $\mathcal{L}_\delta : L^1(\mathbb{S}^1, \mathbb{R}) \to L^1(\mathbb{S}^1, \mathbb{R})$ associated to the extended system $(\mathbb{S}^1, T, h)$ as
$$\mathcal{L}_\delta(\phi) = L_{\delta,\phi}(\phi) \tag{17}$$
for each $\phi \in L^1(\mathbb{S}^1, \mathbb{R})$.

Coup. maps, the dynamics

- Hence the measure representing the current state of the system determines the measure which represents the next state of the system, defining a function between probability measures

$$\mu \longrightarrow L_{\Phi_{\delta,\mu}\circ T}(\mu).$$

- More in general one can consider a family of transfer (Markov) operators $L_{\delta,\nu} : SM \longrightarrow SM$ depending on a probability measure ν and δ and the associated *self consistent transfer operator* $\mathcal{L}_\delta : PM(\mathbb{S}^1) \longrightarrow PM(\mathbb{S}^1)$ defined as

$$\mu \longrightarrow \mathcal{L}_\delta(\mu) := L_{\delta,\mu}(\mu) \tag{18}$$

SeCoTrOp, the notation

$$\mathcal{L}_\delta(\mu) = L_{\delta,\mu}(\mu) \tag{19}$$

- The notation $\mathcal{L}_\delta$ emphasizes the dependence on δ. In the following we will be interested to certain sets of values of δ or the limit $\delta \longrightarrow 0$.
- We use the calligraphic notation $\mathcal{L}$ to denote some operator which is not necessarily linear and the notation L to denote linear operators.