On the statistical stability of families of attracting sets

and the contracting Lorenz attractor

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1 Setting

Ergodic theory in a (tiny) nutshell

Dynamics: $f : M \to M$ **Observable**: $\varphi : M \to \mathbb{R}$

- **Invariant measure**: $\mu(f^{-1}A) = \mu(A)$;
- **Ergodic measure**: $A = f^{-1}A \implies \mu(A) \in \{0, 1\}$.
- **Birkhoff Ergodic Theorem**: if μ is ergodic, then

$$\underline{\text{Time average}} \quad \text{tends to} \quad \underline{\text{space average}}$$

$$\varphi_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \quad \xrightarrow[n\to\infty]{\mu-\text{a.e}} \quad \mathbb{E}(\varphi) = \int \varphi\, d\mu.$$

Continuous time

M smooth Riemannian manifold

$X^t : M \to M$ smooth flow (i.e., $X^{t+s} = X^t \circ X^s$ for $s, t \in \mathbb{R}$)

- **Invariant measure**: $\mu(X^t A) = \mu(A), \ \forall 0 < t \le 1;$
- **Ergodic measure**: $\exists \varepsilon > 0 : A = X^t A, \ \forall 0 < t < \varepsilon \implies \mu(A) \in \{0, 1\}.$
- **Birkhoff Ergodic Theorem**: if μ is ergodic, then

$$\underline{\text{Time average}} \quad \text{tends to} \quad \underline{\text{space average}}$$

$$\varphi_T(y) = \frac{1}{T}\int_0^T \varphi(X^t y)\, dt \quad \xrightarrow[T\to\infty]{\mu-\text{a.e}} \quad \mathbb{E}(\varphi) = \int \varphi\, d\mu.$$

1.1 Physical measures

Physical/SRB measure.

Is there an **invariant** **physical/SRB measure** μ_{SRB}?

That is, a measure μ_{SRB} so that $Leb(B(\mu_{SRB})) > 0$ where

$$B(\mu_{SRB}) = \left\{ y \in M : \frac{1}{T}\int_0^T \varphi(X^t y)\, dt \xrightarrow[T\to\infty]{} \int \varphi\, d\mu_{SRB}, \forall \varphi \in C(M) \right\}$$

is the ***ergodic basin*** of μ_{SRB}.

This kind of measure provides asymptotic information on a set of trajectories that one hopes is large enough to be observable in real-world models.

1.2 Stability

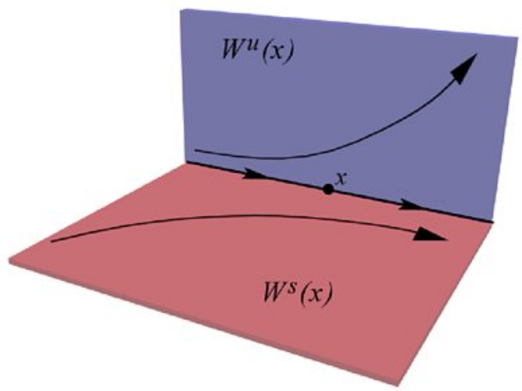
Statistical stability

Can we ***allow for small errors on the formulation of the dynamics not to disturb too much*** the long term behavior?

If we consider

$$\mathcal{U} = \{Y \text{ vector field s.t. } Y^t \text{ admits physical measure } \mu^Y\}$$

and $X \in \mathcal{U}$. Then X is **statistically stable** if



Small errors disturb space average

$$\|Y_n - X\|_{C^1} \xrightarrow[n\to\infty]{Y_n\in\mathcal{U}} 0 \quad \Longrightarrow \quad \int \varphi \, d\mu^{Y_n} \xrightarrow[n\to\infty]{\varphi\in C^0(M)} \int \varphi \, d\mu$$

1.3 Existence

Some known results on existence of physical measures — — ——Hyperbolic versus singular flows

Hyperbolic flows

Hyperbolic flows: all trajectories have a pair of complementary directions:

- in one of them all orbits converge to the trajectory;
- in the other direction all orbits diverge from the trajectory.

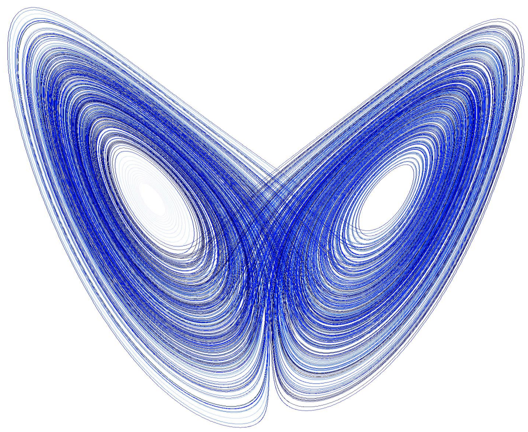
Hyperbolic flows are "classical"

Hyperbolic Theory is the basis for Dynamical Systems Theory: it provides the most mathematically rigorous and deep understanding of an important class of dynamical systems.

This is an *open class* of flows: all flows nearby an hyperbolic flow are also hyperbolic.

Hyperbolic flows do not admit fixed points (singularities or equilibria) accumulated by regular orbits in invariant sets.

However, **there are important open classes of systems which are not hyperbolic** and that frequently appear in applications.



Singular flows which are "almost hyperbolic"

The Lorenz attractor is a flow with an equilibrium accumulated by regular orbits which also belongs to an open class

Attractors and attracting sets

An invariant compact set Λ is an **attracting set** for a vector field X if there exists a **trapping region** U s.t.

$$\overline{X^t(U)} \subset U \text{ for large } t > 0 \text{ and } \Lambda = \bigcap_{t>0} \overline{X^t(U)}.$$

An attracting set becomes an **attractor** if Λ is transitive, that is, we can find $x \in \Lambda$ s.t.

$$\mathcal{O}^+(x) = \{X^t(x) : t > 0\} \text{ is dense in } \Lambda.$$

Existence of physical measures

family vec. fields	physical measures	ergodic basins
Anosov flows (transitive) Axiom A flows **(Hyperbolic)**	unique one for each **attractor**	$\text{Leb}(M \setminus B(\mu)) = 0$ $\text{Leb}(U_i \setminus B(\mu_i)) = 0$ for each attractor
geometric Lorenz attractor	unique	$\text{Leb}(U \setminus B(\mu)) = 0$
contracting Lorenz attractor	unique	$\text{Leb}(U \setminus B(\mu)) = 0$
sectional-hyperbolic attracting sets	finitely many	$\text{Leb}\left(U \setminus \cup_i B(\mu_i)\right) = 0$

Except the contracting Lorenz (Rovella) attrator, all the other families are C^r open families ($r \geq 1$).

Physical measures and equilibrium states

Physical measures and equilibrium states

Let Λ be a **sectional-hyperbolic attracting set / geometrical or contracting Lorenz attractor / hyperbolic attractor** for a C^2 vec. field G with trapping region U. Then the following are equivalent:

1. μ is an equilibrium state with respect to the central jacobian: $h_\mu(X^1) = \int \log|\det DX^1 \mid E^c|\, d\mu > 0$;

2. μ is a SRB measure, i.e., admits an **abs. cont. disintegration along unstable manifolds**;

3. μ is a physical measure, i.e., $\mathrm{Leb}(B(\mu)) > 0$;

Moreover, the family $\mathbb{E}$ of all invariant physical measures is the following convex hull

$$\mathbb{E} = \left\{ \sum_{i=1}^{k} t_i \mu_i : \sum_i t_i = 1; 0 \leq t_i \leq 1, i = 1, \ldots, k \right\}.$$

Entropy expansiveness

Topological entropy

Let $g : M \to M$ be a continuous map and $K \subset M$.

For $\varepsilon > 0$, $n \geq 1$ and $x \in M$

$$B(x, \varepsilon, n) = \{y \in M : d(g^j x, g^j y) < \varepsilon, \quad \forall 0 \leq j < n\}.$$

A subset $E \subset M$ is a (n, ε)**-generator for** K if $\{B(y, \varepsilon, n) : y \in E\}$ is an open cover of K.

Let $r_n(K, \varepsilon)$ be the smallest cardinality of a (n, ε)-generator for K and $r(K, \varepsilon) = \limsup \log r_n(K, \varepsilon)^{1/n}$.

The ***topological entropy of*** g ***on*** K is given by

$$h_{top}(g, K) = \lim_{\varepsilon \to 0} r(K, \varepsilon),$$

and the ***topological entropy of*** g is defined by $h_{top}(g) = h_{top}(g, M)$.

Entropy expansiveness

For $x \in M$ and $\varepsilon > 0$ we define the ***two-sided*** ε***-dynamical ball at*** x as

$$B(x, \varepsilon, \infty) = \{y \in M : d(g^n x, g^n y) < \varepsilon, \forall n \in \mathbb{Z}\}$$

and say that g ***is*** ε***-entropy expansive*** if all these infinite dynamical balls have zero topological entropy, that is,

$$\sup_{x \in M} h_{top}\big(g, B(x, \varepsilon, \infty)\big) = 0.$$

Proposition

If G is entropy expansive, then the metric entropy function $\mu \mapsto h_\mu(X^1)$ is **upper semicontinuous.**

1.4 Conditions

Conditions for statistical stability $----$of families of vector fields

Conditions for statistical stability

Let $\mathcal{G}$ be a collection of vec. fields with a trapping region U, $s \in N \mapsto G_s \in \mathcal{G}$ is a continuous parametrization, and $\Lambda_s(U) = \cap_{t>0}\overline{\phi_t^{G_s}(U)}$ the corresponding attracting set.

Theorem

Assume each Λ_s supports finitely many ergodic physical measures $\mu_i^s, 1 \leq i \leq k_i$ s.t. Leb $\big(U \setminus \sum_i B(\mu_i^s)\big) = 0$; and

1. there are potentials $\psi_s : \Lambda_s \to \mathbb{R}$ s.t. $0 = h_\mu(\phi_1^{G_s}) + \int \psi_s \, d\mu \iff \mu$ a physical measure;
2. the following map is continuous $\Psi : W(U) := \{(s, x) \in N \times U : x \in \Lambda_s(U)\} \to \mathbb{R}$ given by $\Psi(s, x) = \psi_s(x)$; and
3. the family $\mathcal{G}$ is robustly entropy expansive.

Then $G_s \in \mathcal{G}$ is statistically stable.

Statistical stability with several physical measures

In the general setting of the statement we have:

Statistical stability

For each converging sequence $s_n \in N$ to $s \in N$ and every choice μ^{s_n} of a physical measure supported on Λ_{s_n}, **every weak* accumulation point** μ **of** $(\mu^{s_n})_{n\geq 1}$ **is a convex linear combination of the ergodic physical measures of** Λ_s.

More precisely, this means that

1. there are weights $\alpha_i \geq 0$ such that $\sum_i \alpha_i = 1$ and $\mu = \sum_i \alpha_i \mu_i^s$; and
2. we have $\left|\int \varphi \, d\mu^{s_n} - \sum_i \alpha_i \int \varphi \, d\mu_i^s\right| \xrightarrow[n\to\infty]{} 0$ for every continuous observable $\varphi : U \to \mathbb{R}$.

2 Examples

Known examples of application

2.1 Hyperbolic

Axiom A (hyperbolic) flows
Attracting sets for an Axiom A vector field G of class C^2 are finite unions of hyperbolic attractors (basic pieces) which admit a unique physical measure that is also the unique eq. state w.r.t. $\psi_G = -\log|\det D\phi_1^G \mid E^u|$ (**here ϕ_t^G is the flow generated by** G).

Moreover, the hyperbolic property is robust and so the map $X \mapsto \psi_X$ is well defined for vector fields X close to G in the C^2 topology.

In addition, each X is a neighborhood of G is not only hyperbolic but also entropy expansive.

Hence, we may apply the Main Theorem and deduce statistical stability for each basic piece of an Axiom A vector field which is an attractor.

2.2 Singular-hyperbolic

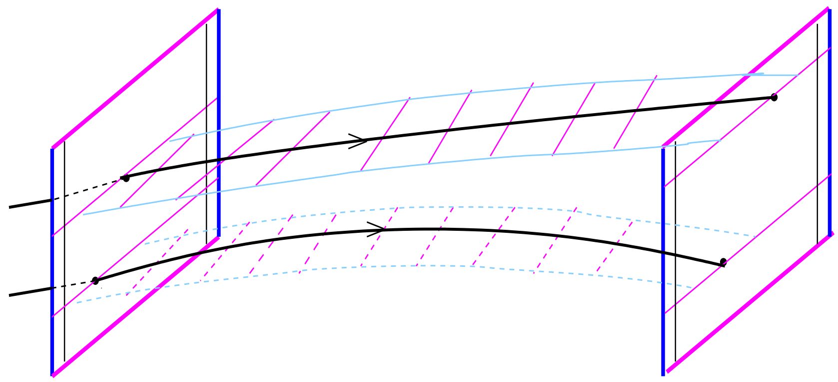
Singular-hyperbolic attracting sets – encompassing the geometrical Lorenz attractor
Singular-hyperbolicity is an extension of the notion of hyperbolicity encompassing sets with equilibria accumulated by regular orbits.

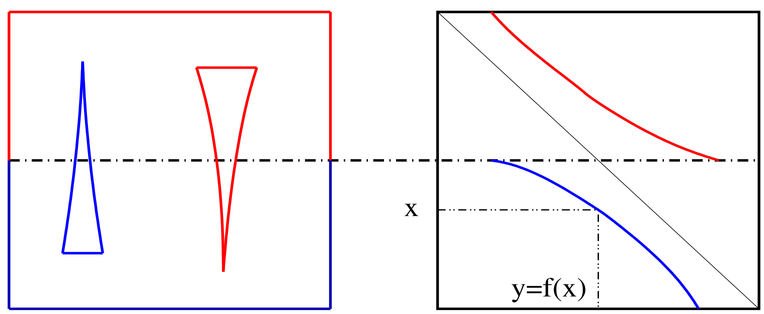
A **singular-hyperbolic set** Λ is

- a *partially hyperbolic set*: there exists a splitting $T_\Lambda M = E^s \oplus E^{cu}$, where $d_s = \dim E_x^s \geq 1$ and $d_{cu} = \dim E_x^{cu} = 2$ for $x \in \Lambda$, and constants $C > 0$, $\lambda \in (0,1)$ s.t. for $t > 0$ we have
  - *uniform contraction on E^s*: $\|D\phi_t|E_x^s\| \leq C\lambda^t$; and
  - *domination*: $\|D\phi_t|E_x^s\| \cdot \|D\phi_{-t}|E_{\phi_t x}^{cu}\| \leq C\lambda^t$.
- with *area expansion on E^{cu}*: $|\det(D\phi_t \mid E_x^{cu})| \geq C\lambda^{-t}$;
- any equilibrium of Λ, if any, is hyperbolic.

Singular-hyperbolic attracting sets and statistical stability
The assumptions of the Main Theorem are known to hold for singular-hyperbolic attracting sets with the potential $\psi_G = -\log|\det D\phi_1^G \mid E^{cu}|$: **existence of finitely many physical/SRB measures** and **robust entropy expansiveness** for singular-hyperbolic attracting sets is established in

- A., **M. J. Pacifico**, Pujals, Viana: "Singular-hyperbolic attractors are chaotic". TAMS, 2009. [unique SRB, transitive case]





- A., Souza, Trindade: “Upper Large Deviations Bound for Singular-Hyperbolic Attracting Sets”. JDDE, 2019. [finite # erg. SRB, non-transitive]
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Other strategies to obtain statistical stability

The dynamics on singular-hyperbolic attracting sets is amenable to **reduction to a global Poincaré return map on a finite collection of cross-sections**:

Reduction to one-dimensional transformation

There is also a further reduction to **a quotient map along the stable leaves tangent to the stable bundle**.

For the geometric **Lorenz attractor, this procedure ends with the one-dimensional Lorenz transformation**.

Statistical properties from the reduction

The physical measure can be constructed from the acip of the one-dimensional map and statistical stability can be deduced from this:

- Alves, Soufi: “Statistical stability of geometric Lorenz attractors”. Fund. Math., 2014.
- Bahsoun, Ruziboev: “On the statistical stability of Lorenz attractors with a $C^{1+\alpha}$ stable foliation”. ETDS, 2018.

Many other finer properties can be deduced:

- A., Melbourne: “Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor”. AHP, 2016.
- Bahsoun, Melbourne, Ruziboev: “Variance Continuity for Lorenz Flows”. AHP, 2020

New examples of application

2.3 Sectional-hyperbolic

Sectional-hyperbolic flows

Sectional-hyperbolicity is an extension of singular hyperbolicity with central dimension $d_{cu} > 2$ where the area expansion property is replaced by ***sectional expansion***: there are $K, \theta > 0$ s.t. **for every two-dimensional subspace** $P_x \subset E^{cu}_x$

$$|\det(D\phi_t \mid P_x)| \geq K e^{\theta t} \quad \text{for all } x \in \Lambda, t \geq 0.$$

Theorem (A., ETDS, 2021)

Every sectional-hyperbolic attracting set for a C^2 vector field G admits finitely many $\mu_1, \dots, \mu_k$ ergodic physical/SRB measures s.t. $h_{\mu_i}(\phi_1^G) + \int \psi_G \, d\mu_i = 0$ and Leb $\big(U \setminus \sum_i B(\mu_i)\big) = 0$.

Recall that $\psi_G = -\log|\det D\phi_1^G \mid E^{cu}|$.

Statistical stability for sectional-hyperbolic attracting sets

Since

- **sectional-hyperbolicity is a C^1 open property**, then the family of vector fields having sectional-hyperbolic attracting sets is C^1 open; and
- **entropy expansiveness** was already obtained by
  - **M. J. Pacifico**, F. Yang, J. Yang: “Entropy theory for sectional hyperbolic flows”. An. l’IHP, 2020.

then we have **all the conditions for statistical stability for sectional-hyperbolic attractors**.

Contracting Lorenz family of attractors $----$ also known as “Rovella attractors” $----$ which is NOT AN OPEN FAMILY of vector fields

2.4 Contracting Lorenz

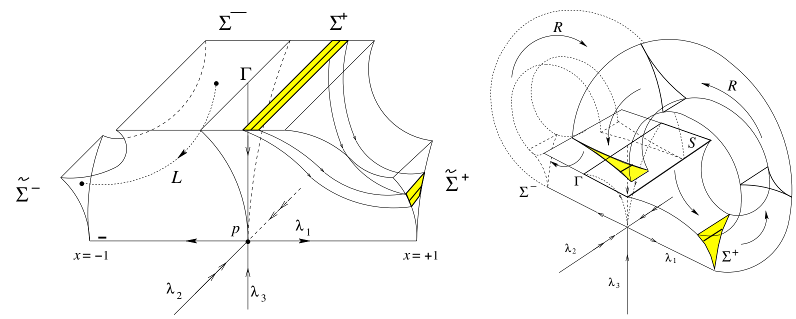
The Rovella family of attractors

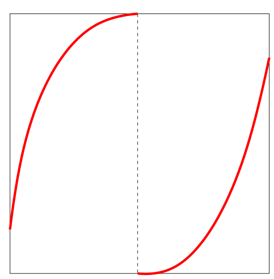
This is a **modification of the geometric Lorenz attractor – the are expanding direction at the equilibrium is replaced by an area contracting direction**: start with a linear vector field $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$ in $[-1, 1]^3$ with real eigenvalues at the singularity s.t.

$$-\lambda_2 > -\lambda_3 > \lambda_1 > 0, \quad r = -\frac{\lambda_2}{\lambda_1}, \quad s = -\frac{\lambda_3}{\lambda_1}, \quad \text{and} \quad r > s + 3.$$

Note that $\lambda_1 + \lambda_3 < 0$ **while in the geometric Lorenz attractor the construction starts with** $\lambda_1 + \lambda_3 > 0$.

Geometric construction and quotient map





Smooth foliation and quotienting

The condition $r > s + 3$ ensures the existence of a C^3 **uniformly contracting stable foliation for the Poincaré first return map of all small enough perturbations of the contracting geometric Lorenz flow**.

Using this, we write the Poincaré first return map as $R_0(x, y) = (T_0(x), H_0(x, y))$, where $H_0(x, y)$ uniformly contracts distances along y and

1. $T_0 : [-1/2, 1/2] \setminus \{0\} \to [-1/2, 1/2]$ is piecewise C^3 with two onto branches s.t. $T_0'(x) = O(x^{s-1})$ at $x = 0$;

2. $T_0(0^+) = -1/2$ and $T_0(0^-) = +1/2$;
3. $T_0' > 0$ on $[-1/2, 1/2] \setminus \{0\}$;
4. $\pm 1/2$ are preperiodic repelling for T_0.

Family of 2-almost persistent attractors

Rovella (Bull. Braz. Math. Soc., 1993) showed that the flow of this vector field G_0 has an attractor $\Lambda_0 = \cap_{t>0}\overline{\phi_t^{G_0}(U)}$ and **its perturbations admit a two-parameter family of vector fields which is "almost persistent"**, as follows.

There exists a 2-dimensional C^∞ submanifold N of C^3 vector fields $\mathfrak{X}^3(\mathbb{R}^3)$ containing G_0 s.t. the subset $S \subset N$ corresponding to an attractor $\Lambda_{G_s} = \cap_{t>0}\overline{\phi_t^{G_s}(U)}$ for each $s \in S$, then

$$\lim_{r\to 0} \frac{\mathrm{Leb}(B_r(x) \cap S)}{\mathrm{Leb}(B_r(x))} = 1,$$

where $B_r(x)$ is an r-ball in N and Leb is the area measure.

Persistent asymptotic sectional-hyperbolicity

Theorem (Vivas, San Martin: Nonlinearity, 2020)
The attractor Λ_0 is 2-dimensionally almost persistent **asymptotically sectional hyperbolic** in the C^3 topology.

A compact invariant partially hyperbolic set Λ of a vector field G, with $d_{cu} = 2 = \dim E^{cu}$, whose singularities are hyperbolic, is *asymptotically sectional hyperbolic* if there exists $c_* > 0$ so that

$$\limsup_{T\to\infty} \frac{1}{T} \log|\det(D\phi_T \mid E_x^{cu})| \geq c_*, \quad \forall x \in \Lambda \setminus \bigcup_{\sigma\in\Lambda\cap\mathrm{Sing}(G)} W^s(\sigma).$$

Statistical stability of the Rovella family

Theorem (A., JSP, 2021)
The family $\mathcal{G}$ of contracting Lorenz attractors, with trapping region U, is such that each of its elements admits a unique physical measure, whose basin covers U except for zero Leb-measure subset and is statistically stable.

The existence of the unique physical/SRB measure μ_a for each $G_a \in \mathcal{G}$ follows from the fact that

- the Poincaré map $R_a(x, y) = (T_a(x), H_a(x, y))$ satisfies
  - $H_a(x, \cdot)$ is a uniform contraction;
  - T_a is a one-dimensional non-unif. exp. map with slow recurrence to the discont. critical point $\{0\}$

  then *every ergodic acip ν_a w.r.t. T_0 induces μ_a which is an ergodic hyperbolic SRB-measure w.r.t. G_a on Λ_a.*

The physical/SRB measure in the Rovella family

This ensures, by well-knonw arguments, that μ_a **admits an absolutely continuous disintegration along unstable manifolds** and is an **ergodic physical measure**.

Moreover, since the flow direction on partially hyperbolic sets is contained in the central-unstable direction, then Oseledets' Theorem ensures

$$\int \log |\det(D\phi_1^{G_a} \mid E^{cu})| \, d\mu_a = \lambda^+(x) \geq c_* > 0,$$

where $\lambda^+(x) = \lim_{T\to\infty} \log|\det(D\phi_T \mid E_x^{cu})|^{1/T}$ is the largest Lyap. exponent along the two-dimensional bundle E^{cu} for μ_a-a.e. x. Hence (Ledrappier-Young characterization)

$$h_{\mu_a}(\phi_1^{G_a}) = \int \log |\det(D\phi_1^{G_a} \mid E^{cu})| \, d\mu_a > 0.$$

Robust expansiveness

Denote by $S(\mathbb{R})$ the set of surjective increasing continuous functions $h : \mathbb{R} \to \mathbb{R}$. The flow is ***expansive* on an invariant compact set** Λ if for every $\varepsilon > 0$ there is $\delta > 0$ s.t. for any $h \in S(\mathbb{R})$ and $x \in \Lambda$

$$d(\phi_t(x), \phi_{h(t)}(y)) \leq \delta, \forall t \in \mathbb{R} \implies \\ \implies \exists t_0 \in \mathbb{R} \text{ s.t. } \phi_{h(t_0)}(y) \in \phi_{[t_0-\varepsilon, t_0+\varepsilon]}(x).$$

$\mathcal{G}$ is ***robustly expansive*** on $\Lambda_s = \cap_{t>0}\overline{\phi_t^{G_s}(U)}$, $s \in N \cap S$, if $\exists$ nghbhd. V of s in N s.t. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for $u \in V \cap S$, $x \in \Lambda_u = \cap_{t>0}\overline{\phi_t^{G_u}(U)}$ and $h \in S(\mathbb{R})$, then

$$d(\phi_t^{G_u}(x), \phi_{h(t)}^{G_u}(y)) \leq \delta, \forall t \in \mathbb{R} \implies \\ \implies \exists t_0 \in \mathbb{R} \text{ s.t. } \phi_{h(t_0)}^{G_u}(y) \in \phi_{[t_0-\varepsilon, t_0+\varepsilon]}^{G_u}(x).$$

Robust entropy expansiveness from robust expansiveness

Proposition (Bowen, 1972)

A robustly expansive attracting set $\Lambda_G(U)$ on a family $\mathcal{G} : N \to \mathfrak{X}^r(M)$ admits $\delta > 0$ which is a constant of h-expansiveness for each flow in the family.

To fulfill all the conditions of statistical stability, it is enough to obtain

Lemma (Robust expansiveness for Rovella attractors)

The family $\mathcal{G}$ of Rovella attractors is robustly expansive.

Robust expansiveness for Rovella attractors

This is a consequence of the **locally eventually onto** property as follows. We write $c_a^\pm = T_a(0^\pm) = \lim_{t\to 0^\pm} f(t)$ and note that $c_a^- < 0 < c_a^+$ and $c_a^\pm \to \pm 1/2$ when $a \to 0$.

Lemma l.e.o. (Lemma 4.1 in Metzger: An. l'IHP, 2000)
There exists a C^3 neighborhood $\mathcal{V}$ of G_0 so that if $G_a \in \mathcal{V}$, then the map T_a is locally eventually onto, that is, for any interval $J \subset [-1/2, 1/2] \setminus \{0\}$ there exists $n = n(J) > 0$ so that $f^n(J) \subset [c_a^-, c_a^+]$.

Consequently, ***there does not exist a pair of points*** $x_0 < y_0$ ***with the same sign in*** $[-1/2, 1/2] \setminus \{0\}$ ***so that*** $T_a^n[x_0, y_0]$ ***does not contain the origin for all*** $n \geq 1$**.**

3 Proof

3.1 Statistical Stability

Proof of statistical stability$----$ Consequence of continuity of equilibrium states

3.2 Continuity of equilibrium states

Argument for statistical stability

Theorem (Continuity of equilibrium states)
Let $f : X \times M \to M$ and $\psi : X \times M \to \mathbb{R}$ be continuous, with $X = N \cap S$, s.t.

1. f_a admits some equilibrium state for ψ_a: $\exists \mu_a \in \mathcal{P}_{f_a}(M)$ s.t. $P_{f_a}(\psi_a) = h_{\mu_a}(f_a) + \int \psi_a \, d\mu_a$ for all $a \in X$.
2. For each weak* accumulation point μ_0 of μ_a when $a \to * \in X$, let $a_k \to *$ s.t. $\mu_k = \mu_{a_k} \to \mu_0$, write $f_k = f_{a_k}$, $\psi_k = \psi_{a_k}$ and assume also that
   (a) there exists a finite Borel partition ξ of M such that $h_{\mu_k}(f_k) = h_{\mu_k}(f_k, \xi)$ for all $k \geq 1$; and $\mu_0(\partial \xi) = 0$.
   (b) $P_{f_k}(\psi_k) \to P_{f_*}(\psi_*)$ when $k \to \infty$.

Then every weak* accumulation point μ of $(\mu_k)_{k\geq 1}$ when $k \to \infty$ is a equilibrium state for f_* and the potential ψ_*.

Entropy expansiveness and generating partitions

Theorem (Bowen, 1972)
Let M be a compact metric space of finite dimension, $\varepsilon > 0$ an h-expansiveness constant for $f : M \to M$, and ξ a Borel partition of M with $\operatorname{diam}(\xi) < \varepsilon$. Then $h_\mu(f) = h_\mu(f, \xi)$ for each f-inv. prob. masure μ.

We may now use the robust entropy expansiveness to build a uniform generating partition ξ satisfying the conditions for continuity of equilibrium states, together with the assumption

$$h_{\mu_a}(\phi_1^{G_a}) - \int \log|\det(D\phi_1^{G_a} \mid E^{cu})|\, d\mu_a = 0$$

to apply the Theorem on Cont. of Eq. States with $\psi_a = -\log|\det(D\phi_1^{G_a} \mid E^{cu})|$.

THE END.

OBRIGADO!

PARABÉNS ZÉZE!

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